1 Roughly this is the staff part, we can shuffle them

1. $$\int_1^2 \frac{1}{\log_x(2)}dx$$

   $(\ln(4)-1)/\ln(2)$, via $\log_x(2)=\ln(2)/\ln(x)$

2. $$\int_0^\pi \frac{8dx}{5+2\cos x}$$

   $8\pi/\sqrt{21}$. Two ways to arrive at this. Weierstrass substitution is straightforward. Can also be done by noting it is half the integral from 0 to 2π and setting $z=e^{ix}$ and then using the residue theorem. Stolen from CA for physics so don't give this to Tijmen/Tibboel

3. $$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+729}dx$$

   First note $729=27^2$

   - Using the residue theorem: this is the real part of $e^{iz}/(729+z^2)$. Via Jordan's Lemma we can instead compute the integral over the semicircular contour from $-r$ to r as $r\to\infty$. The only pole contained in that contour is $i27$ which is simple with residue $-ie^{-27}/54$. Hence via the residue theorem the integral equals $\pi e^{-27}/27$
   - Using Feynman's Trick: Introduce an extra parametre t to obtain $I(t)=\int_{-\infty}^{\infty}\frac{\cos(tx)}{x^2+729}dx$, then $\partial_t I(t)=-\int_{-\infty}^{\infty}\frac{x^2\sin(tx)}{x(x^2+729)}$. Add $0=729-729$ to x^2 in the numerator to arrive at

     $$I'(t)=-\int_{-\infty}^{\infty}\frac{(x^2+729)\sin(tx)}{x(x^2+729)}dx+\int_{-\infty}^{\infty}729\frac{\sin(tx)}{x^2+729}dx$$

     The first simplifies to the Dirichlet integral which is known to be π, maybe put that one in as well. Differentiating the second one one more time gives back the original integral: $I''(t)=729I(t)$.

     $$I(0)=\int_{-\infty}^{\infty}\frac{1}{729^2+x^2}dx=\pi/27$$

     So we have a system of differential equations

     $$\begin{cases} I''(t)=729I(t), \\ I(0)=\pi/27, \\ I'(0)=-\pi \end{cases}$$

     this has solution

     $$I(t)=\frac{\pi e^{-27t}}{27}$$

     so that $I(1)=\pi e^{-27}/27$

4. $$\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{12}}dx$$

- Can be done by setting $x = \tan(u)$ turning it into

$$2\int_0^{\pi/2} \cos^{22}(u)du = \frac{1}{22}\sin(u)\cos^{21}(u)|_0^{\pi/2} + \frac{21}{22}\int_0^{\pi/2} \cos^{20}(u)du$$

  where the first term is 0, so we obtain

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{12}} = \frac{21 \cdot 19 \cdot \dots \cdot 3}{22 \cdot 20 \cdot \dots \cdot 6}\int_0^{\pi/2} \cos^2(x) = \frac{1}{(1+x^2)^{12}} = \frac{21 \cdot 19 \cdot \dots \cdot 3}{22 \cdot 20 \cdot \dots \cdot 6}\frac{\pi}{4}$$

- Via the residue theorem. Via Jordan's lemma we can integrate over the semicircular contour of radius r instead. This has one pole of order $n+1$ at i. Computing its residue means computing

$$\left.\frac{1}{11!}\frac{d^{11}}{dz^{11}}\frac{1}{(z+i)^{12}}\right|_{z=i} \cdot 2\pi i$$

  coming out to the same as earlier.

5.

$$\int_{-\infty}^{\infty} \frac{e^{5x-x^2/2}}{\sqrt{2\pi}}dx$$

The silly way of doing this is recognising it as the moment generating function of an $N(0,1)$ distribution evaluated at 5, making it $e^{25/2}$. Make sure to give this one to Marco. Can also be done by some horrid substitutions on the standard Gaussian integral.

6.

$$\int_0^{\infty} e^{-42x}\cos(69x)$$

I suppose this is possible to do with integration by parts, but here are two more interesting methods:

- Consider the subspace $V = \operatorname{span}\{e^{-42x}\cos(69x), e^{-42x}\sin(69x)\}$, and consider $D : V \to V$ the differentiation operator, i.e.

$$\begin{aligned} De^{-42x}\cos(69x) &= -42e^{-42x}\cos(69x) - 69e^{-42x}\sin(69x), \\ De^{-42}\sin(69x) &= 69e^{-42}\cos(69x) - 42e^{-42}\sin(69x) \end{aligned}$$

  or in matrix notation:

$$D = \begin{bmatrix} -42 & 69 \\ -69 & -42 \end{bmatrix}$$

  Do not tell your analysis teacher, but $D^{-1} = \int$, hence via Cramer's rule

$$\int = \frac{1}{42^2 + 69^2}\begin{bmatrix} -42 & -69 \\ 69 & -42 \end{bmatrix}$$

  Hence the coefficients of the indefinite integral are

$$\frac{1}{42^2 + 69^2}\begin{bmatrix} -42 & -69 \\ 69 & -42 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{42^2 + 69^2}\begin{bmatrix} -42 \\ 69 \end{bmatrix}$$

  Hence

$$\int_0^{\infty} e^{-42x}\cos(69x)dx = \left.\frac{-42\cos(69x) + 69\sin(69x)}{42^2 + 69^2}e^{-42x}\right|_0^{\infty} = \frac{42}{42^2 + 69^2}$$

  simplifiying to 14/2175

- Recognise the laplace transform to conclude the same.
- Bonus silly way, can be done by integrating $e^{-\sqrt{42^2+69^2}z}$ over an appropriate contour, so Stein-Shakarchi assures me. This involves doing actual calculus so that's too scary.

7.
$$\int_0^\infty \frac{x}{e^x-1}dx$$

$$\int_0^\infty \frac{x}{e^x-1}\,\mathrm{d}x = \int_0^\infty \frac{xe^{-x}}{1-e^{-x}}\,\mathrm{d}x = \int_0^\infty xe^{-x}\sum_{n=0}^\infty e^{-nx}\,\mathrm{d}x$$
$$= \sum_{n=0}^\infty \int_0^\infty xe^{-(1+n)x}\,\mathrm{d}x = \sum_{n=0}^\infty \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

8.
$$\int_0^\infty x^2e^{-x^2}dx$$

$$\left(\int_0^\infty x^2e^{-x^2}\,\mathrm{d}x\right)^2 = \int_0^\infty x^2e^{-x^2}\,\mathrm{d}x\int_0^\infty y^2e^{-y^2}\,\mathrm{d}y = \iint_{[0,\infty)^2} x^2y^2e^{-(x^2+y^2)}dA$$

Switch to polar coordinates

$$= \int_0^{\pi/2}\int_0^\infty r^5\cos^2\theta\sin^2\theta e^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/2}\cos^2(\theta)\sin^2(\theta)\,\mathrm{d}\theta\int_0^\infty r^5e^{-r^2}\,\mathrm{d}r$$

the first can be evaluated by considering $\cos^2\alpha = (1-\sin^2\theta)$, so and $\sin^2(x) = (1-\cos(2x))/2$, so

$$\int_0^{\pi/2}\sin^2\theta(1-\sin^2\theta)\,\mathrm{d}\theta = \frac{1}{8}\int_0^{\pi/2} 1-\cos(4x)\,\mathrm{d}x = \pi/16$$

the second can be done through substituting $u = r^2$, being equal to 1, hence the integral is $\sqrt{\pi/16} = \sqrt{\pi}/4$

9.
$$\int_0^1 \sin(\cos^{-1}(x))dx$$

Note $\sin(\theta) = \sqrt{1-\cos^2(\theta)}$ on this interval, hence the integrant equals $\sqrt{1-x^2}$, which is just the area of a quarter cicle, being $\pi/4$.

10.
$$\int_0^{\pi/2}\log\cos(x)dx$$

Substitute $u = \pi/2 - x$ to see it equals $\int_0^{\pi/2}\log\sin(u)du$, meaning our integral equals

$$\begin{aligned} I &= \int_0^{\pi/2}\log(2\cos(x/2)\sin(x/2))dx \\ &= \int_0^{\pi/2}\log(2)+\log(\cos(x/2))+\log(\sin(x/2)) \\ &= \pi/2\log(2) + \int_0^{\pi/4}\log\cos(x)+\log\sin(x)dx \end{aligned}$$

One notes $\int_0^{\pi/4} \log\sin(x)dx = 1/2\int_{\pi/2}^{\pi/4} \log\sin(x)$ via the same substitution as before, hence

$$I = \pi/2\log(2) + 2I \iff I = -(\pi/2)\log(2)$$

11.
$$\int_0^2 \{x^2\}dx, \quad \text{With } \{x\} \text{ denoting the fractional part of } x, \text{ e.g.}\{1.5\} = 0.5$$

The function equals x^2 on $[0,1]$, x^2-1 on $[1,\sqrt{2}]$, x^2-2 on $[\sqrt{2},\sqrt{3}]$, and x^2-3 on $[\sqrt{3},2]$, so we obtain

$$\int_0^1 x^2dx + \int_1^{\sqrt{2}} x^2 - 1dx + \int_{\sqrt{2}}^{\sqrt{3}} x^2 - 2dx + \int_{\sqrt{3}}^2 x^2 - 3dx$$
$$= 1/3 + 1/3(2-\sqrt{2}) + 4/3\sqrt{3} - \sqrt{3} + 2\sqrt{3} - 10/3 = -7/3 + \sqrt{2} + \sqrt{3}$$

12.
$$\int_0^{\pi/4} \tan(x)\sec^5(x)$$

Rewrite

$$\int_0^{\pi/4} \frac{\sin(x)}{\cos^5(x)} = -\int_1^{\sqrt{2}/2} 1/\cos^5(x)d\cos(x) = \sec^4(x)/5|_{\cos(x)=\sqrt{2}/2}^{\cos(x)=1} = 1/5(4\sqrt{2}-1)$$

13.
$$\int_0^1 (\log(1/x))^{25}dx$$

25! equivalent of the Γ function by substituting $t = -\log x$.

14.
$$\int_0^\infty e^{-\sqrt[3]{x}}dx$$

Via substitution $u = \sqrt[3]{x}$ we get

$$= 3\int_0^\infty e^{-u}u^2du = 3\cdot 2$$

1.1 Tiebreakers

We reuse some integrals from the qualifier:

1.
$$\int_0^\pi \max(\cos(x), \sin(x))$$

$1+\sqrt{2}$, note $\cos(x) \geq \sin(x)$ on $[0,\pi/4]$ and $\sin(x) \geq \cos(x)$ on $[\pi/4,\pi]$

2.
$$\int_0^1 \frac{1}{e^{-x}+e^x}$$

$\pi/4$, multiply numerator and denominator by e^x and substitute $u = e^x$, recognise arctan.

3.
$$\int_0^1 \frac{1}{x^2+x+1}dx$$

$\pi/(3\sqrt{3})$

2 Students

1. $$\int_{-1}^{1} \sqrt{x^6} dx$$

   Hasty students might think this is 0 because $\sqrt{x^6} = x^3$ which is odd, but no! in actuality $\sqrt{x^6} = |x|^3$ hence the integral equals $x|x|^4/4$ (one may think this needs to be done piecewise but one would be wrong) hence we obtain $|1|^4/4 - (-1)| - 1|^3/4 = 1/2$.

2. $$\int_{0}^{\pi/2} \frac{1}{1 + \cot^4(x)}$$

   $\pi/4$, follows by the exact same phase-shift trick (King's Property) as the infamous $\frac{1}{1+\tan^n(x)}$. Do not give to Tijmen.

3. $$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{e^{-x} + 1}$$

   Can also be done using King's property. Can also note

   $$\frac{1}{e^{-x}+1} = \frac{e^{x/2}}{e^{-x/2} + e^{x/2}} - \frac{e^{x/2}/2 + e^{-x/2}/2}{e^{-x/2} + e^{x/2}} + 1/2 = \frac{1}{2}\frac{e^{-x/2} - e^{x/2}}{e^{x/2} + e^{-x/2}} + 1/2$$

   where the first term is odd, and will still be upon multiplication by an even function; so it cannot contribute to the integral, hence the integral just equals 1/2 the guassian integral, being $\sqrt{\pi}/2$

4. The tijmen special:

   $$\int_{-\infty}^{\infty} \frac{1}{x^2 - 1 + 1/x^2}$$

   Note the denominator equals $1/(1 + (x - 1/x)^2$. The fact that if both are defined then the Cauchy-Schlömilch transformation $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x - 1/x)$ turns it into $\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \pi$ Can also be done using partial fractions, in this case we get

   $$-\frac{1}{2\sqrt{3}} \frac{x}{-x^2 + \sqrt{3}x - 1} - \frac{1}{2\sqrt{3}} \frac{x}{x^2 + \sqrt{3}x + 1}$$

5. $$\int_{0}^{1} \frac{x^4(1-x)^4}{1+x^2}$$

   $22/7 - \pi$, guessing 0 gives half points. Can be done via long division to write

   $$x^4(1-x)^4 = (1+x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 - 4) + 4$$

   in particular this shows $22/7 > \pi$ as we are integrating a nonnegative function.

6. $$\int_{0}^{\pi} \min(\cos(x), \sin(x)) e^{\max(\cos(x), \sin(x))} dx$$

   The minimum and maximum switch at $\pi/4$ hence

   $$= \int_{0}^{\pi/4} \sin(x) e^{\cos(x)} dx + \int_{\pi/4}^{\pi} \cos(x) e^{\sin(x)} dx$$

   these are easily solved using substitution to be $e - e^{1/2\sqrt{2}}$ and $1 - e^{1/2\sqrt{2}}$ respectively, coming out to be $1 + e - 2e^{1/2\sqrt{2}}$

7.
$$\int_0^\pi \cos^4 x - \sin^4 x dx$$

It follows as

$$\cos^4 x = \cos^2 x(1 - \sin^2 x)$$
$$\sin^4 x = \sin^2 x(1 - \cos^2 x)$$
$$\therefore \cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x = \cos(2x)$$

hence

$$I = \sin(2x)/2\Big|_0^\pi = 0$$

8.
$$\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} \frac{\csc(x + 1/x)\ln x}{x} dx$$

The csc is a red herring, in reality any integral $\int_{1/y}^{y} f(x + 1/x)\ln x/x = 0$ for f continuous: just substitute $u = 1/x$ then you get back your integral with a minus sign.

9.
$$\int_0^1 \sin(\ln(x))dx$$

Substitute $u = \ln x$ to obtain

$$\int_{-\infty}^0 e^u \sin u du$$

which is seen to be $-1/2$ via integration by parts.

10.
$$\int_{-\pi}^{\pi} \cos(6x)\cos(5x)dx$$

0, these are orthogonal in the L^2 norm.

11.
$$\int_0^{1/6} (1 + 6x)(2 - x + 30x^2 - 216x^3 + 1296x^4) + (-5x + 6x^2 - 1)(1 + 6x)dx$$

integrand simplifies to $1 + (6x)^5$, giving

$$1/6 + \frac{6}{6}6^5(1/6)^6 = 1/6 + 1/36 = 7/36$$

12.
$$\int_{\pi/4}^{\pi/3} (\ln(\tan(x) + x\csc(x)\sec(x))\tan(x)^x dx$$

$3^{\pi/6} - 1$, substitute $u = \tan(x)^x$

13.
$$\int_0^1 \frac{\ln(\frac{1}{1-x})}{x} dx$$

Substitute the taylor series for $\ln(1 - x)$, which converges uniformly for $|x| < 1$. Ends up being $\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$.

14.

$$\int_{\pi/3}^{\pi/2} \frac{\csc x}{\sin(x)+1} dx$$

Write out and use Weierstrass substitution, after which use partial fractions. Bounds are slightly ugly, final answer is $-2+\sqrt{3}+\frac{\log(3)}{2}$

15.

$$\int_{-\infty}^{\infty} \sqrt{1+x^2}\sec(x)\tan(x^3)$$

0, odd function

3 Grand Finale

1.

$$\int_0^{\infty} \sin(x^2/2)\cos(x^2/2) dx$$

Integrant equals $\sin(x^2)/2$, which is half the fresnel integral and hence equals $\sqrt{\pi/2}/4$

2.

$$\int_0^{\pi} \frac{1}{\sin(x) - 2\csc(x)} dx$$

$-\pi/2$, multiply numerator and denominator with $\sin(x)$ and recognise arctan.

3.

$$\int_0^{\infty} \frac{\sin(x)\cos(x)}{\sin^4(x)+\cos^4(x)} dx$$

Solution:

$$\frac{\sin(x)\cos(x)}{\sin^4(x)+\cos^4(x)} = \frac{\tan(x)(\tan(x))'}{\tan^4(x)+1}$$

Set $u=\tan(x)$, which integrates to $\tan(x^2)/2$, so the answer is $\pi/4$.

4 Tie-Breakers

1.

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-(nx)^2} \cos\sqrt{x^2+\pi^2} dx$$

$\cos(\pi)$, delta function

2.

$$\int_0^{\infty} \frac{1}{\sqrt{e^x - 1}} \mathrm{d}x$$

π, substitute $u=e^x$, the antiderivative is $\arctan\sqrt{e^x-1}$

3.

$$\int_0^{\infty} \frac{\ln(x)}{1+x^2}$$

0, substitute $u=1/x$.